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Asymptotics for Global Measures of Accuracy of Splines

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We obtain central limit theorems for the stochastic parts of L_p-norms of smoothed cubic spline estimators. The proofs are based on the observation that the variance term of the cubic spline is approximately of a form corresponding to a kernel estimator.

1. INTRODUCTION

We consider the asymptotics of global measures (L_p-norms, $1 \leqslant p < \infty$) of the estimation of a regression function by a cubic smoothing spline. The observations

$$y_i = g(x_i) + \xi_i, \qquad 1 \leqslant i \leqslant n,$$

are given, where the design points x_i, $1 \leqslant i \leqslant n$, are known and evenly spaced on $[a, b]$. For convenience we assume that $x_i = a + (i/n)(b-a)$, $1 \leqslant i \leqslant n$. We assume throughout this paper that the random errors $\{\xi_i, 1 \leqslant i \leqslant n\}$ are independent identically distributed random variables (i.i.d.r.v.'s) satisfying

$$E\xi_i = 0, \qquad 0 < \sigma^2 = E\xi_i^2 < \infty \tag{1.1}$$

and

$$E|\xi_i|^\nu < \infty \qquad \text{for some} \quad \nu > 2. \tag{1.2}$$

The cubic spline estimator g_n of the regression curve g is defined to be a function minimizing

$$\lambda \int_a^b (u''(t))^2\, dt + \frac{1}{n} \sum_{i=1}^n (y_i - u(x_i))^2. \tag{1.3}$$

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The parameter $\lambda = \lambda(n) > 0$ is the smoothing parameter. Splines play a very important role in numerical analysis and stochastic curve estimation, and therefore several authors have studied their properties. For a review and bibliography on splines we refer to De Boor [5], Silverman [18], and Eubank [7].

It is well known that g_n is a linear function of the observations y_i, $1 \leqslant i \leqslant n$, which means that there exists a weight function $G_n(t, x)$ such that

$$g_n(t) = \frac{1}{n} \sum_{i=1}^{n} G_n(t, x_i)\, y_i. \tag{1.4}$$

Silverman [17] studied the connection between spline smoothing and kernel (or convolution or moving average) smoothing. He showed that, under suitable conditions, the weight function G_n is approximately of a form corresponding to smoothing by a kernel function K. The kernel K is given by

$$K(u) = \frac{b-a}{2} \exp(-2^{-1/2} |u|) \sin\left(2^{-1/2} |u| + \frac{\pi}{4}\right). \tag{1.5}$$

As it is usual in curve estimation we write

$$g_n(t) - g(t) = \hat{g}_n(t) + g_{(n)}(t) - g(t),$$

where

$$\hat{g}_n(t) = \frac{1}{n} \sum_{i=1}^{n} G_n(t, x_i)\, \xi_i$$

is the random error and

$$g_{(n)}(t) - g(t) = \frac{1}{n} \sum_{i=1}^{n} G_n(t, x_i)\, g(x_i) - g(t)$$

is the bias (numerical error). It turns out that the approximation of splines with kernel smoothing works very well for the random error but it does not give acceptable results for the bias. This is very similar to the behavior of Fourier type curve estimators (Horváth [8]).

The main result of this paper is the computation of the limit distribution of

$$\hat{I}_n(p) = \int_a^b |\hat{g}_n(t)|^p\, w(t)\, dt,$$

where w is a non-negative weight function. We assume that w is bounded and it is continuous on $[a, b] \setminus C$, where the Lebesque measure of C is zero. We also study the asymptotics of

$$I_n(p) = \int_a^b |\hat{g}_n(t) + u_n(t)|^p\, w(t)\, dt,$$

where $\{u_n(t), a \leqslant t \leqslant b\}$ is a non-random shift. If $u_n(t) = g_{(n)}(t) - g(t)$ is the bias, then this special case of $I_n(p)$ is denoted by $I_n^*(p)$. The integrated squared error $EI_n^*(2)$ has been a very popular measure of accuracy in curve estimation. Wahba [19], Rice and Rosenblatt [15, 16], and Ragozin [14] gave upper bounds for $EI_n^*(2)$ and $E\hat{I}_n(2)$. Lii [12] obtained central limit theorems for L_2-norm of spline density estimators. To the best of our knowledge this is the only result which gives the asymptotic distribution of a functional of a spline. Devroye and Györfi [6] advovated the L_1-norm as a measure of accuracy in estimation of functions. However, the limit distributions of the L_p-norms of splines are unknown. The limit distributions of the L_p-norms of kernel-density estimators were obtained by Csörgő and Horváth [4], Csörgő *et al.* [3], and Horváth [9]. The limit distributions of L_p-norms are of statistical significance, because goodness-of-fit tests can be based on them. The computation of the order of $I_n(p)$ is not enough in statistical applications.

Before we state our results we list a few conditions and introduce some notations. Let $1 \leqslant p < \infty$ and $r_n(t) = n^{1/2}\lambda^{1/8}u_n(t)$. We assume

C1. $\lambda(n) \to 0$,

C2. $\limsup_{n \to \infty} \sup_{a \leqslant t \leqslant b} |r_n(t)| < \infty$,

C3. $\lim_{n \to \infty} \sup_{a \leqslant t \leqslant b - \lambda^{1/4}} \sup_{0 \leqslant s \leqslant \lambda^{1/4}} |r_n(t) - r_n(t+s)| = 0$,

C4. $\limsup_{n \to \infty} n^{1/v - 1/2}\lambda^{-1/4}(n) < \infty$,

C5. $v > \max(3, 2p-2)$.

We define

$$D^2 = \frac{\sigma^2}{b-a} \int_{-\infty}^{\infty} K^2(u)\, du,$$

$$\alpha(s) = \int_{-\infty}^{\infty} K(t)\, K(t+s)\, dt \Big/ \int_{-\infty}^{\infty} K^2(t)\, dt,$$

$$\varphi(x) = (2\pi)^{-1/2} \exp(-x^2/2),$$

$$\psi(u; x, y) = \frac{1}{2\pi(1-u^2)^{1/2}} \exp\left(-\frac{1}{2(1-u^2)}(x^2 + y^2 - 2uxy)\right),$$

$$m = \int_{-\infty}^{\infty} \int_a^b |Dx + r_n(t)|^p\, w(t)\, \varphi(x)\, dt\, dx$$

$$m_* = D^p \int_a^b w(t)\, dt \int_{-\infty}^{\infty} |x|^p\, \varphi(x)\, dx$$

$$\theta^2 = \int_{R^3} \int_a^b |D^2xy + Dr_n(t)(x+y) + r_n^2(t)|^p\, w^2(t)(\psi(\alpha(u); x, y) - \varphi(x)\, \varphi(y))\, dt\, du\, dx\, dy$$

and

$$\theta_*^2 = D^{2p} \int_a^b w^2(t)\,dt \int_{R^3} |xy|^p \,(\psi(\alpha(u); x, y) - \varphi(x)\,\varphi(y))\,dx\,dy\,du.$$

Now we can state our first central limit theorem.

THEOREM 1. *We assume that* C1, C4, *and* C5 *hold. Then, as* $n \to \infty$, *we have*

$$\{(n(b-a)^{1/4}\,\lambda^{1/4}(n))^{p/2}\,\hat{I}_n(p) - m_*\}/(\theta_*^2\,\lambda^{1/4}(n)(b-a)^{1/4})^{1/2} \xrightarrow{\mathscr{D}} N(0, 1),$$

where $N(0, 1)$ *is a standard normal r.v.*

The next theorem may be of more statistical interest.

THEOREM 2. *We assume that* C1–C5 *hold. Then, as* $n \to \infty$, *we have*

$$\{(n(b-a)^{1/4}\,\lambda^{1/4}(n))^{p/2}\,I_n(p) - m\}/(\theta^2\lambda^{1/4}(n)(b-a)^{1/4})^{1/2} \xrightarrow{\mathscr{D}} N(0, 1),$$

where $N(0, 1)$ *is a standard normal r.v.*

Theorem 2 remains true if C3 fails. In this case θ^2 must be replaced by

$$\int_{R^2} \int_a^b \int_{-(b-t)/\lambda^{1/4}}^{(t-a)/\lambda^{1/4}} |D^2xy + Dxr_n(t) + Dyr_n(t - u\lambda^{1/4}(n)) + r_n(t)\,r_n(t - u\lambda^{1/4}(n))|^p \times w^2(t)(\psi(\alpha(u); x, y) - \varphi(x)\,\varphi(y))\,du\,dt\,dx\,dy.$$

Also, we still have Theorem 2 if C2 changed to

$$\limsup_{n\to\infty} \int_a^b |r_n(t)|^\tau\,w(t)\,dt < \infty$$

with $\tau = \max(3, 2p-2)$. If $p = 2$, then the expected value and variance have simpler forms. Assuming $p = 2$ one can write

$$m(2) = D^2 \int_a^b w(t)\,dt + \int_a^b r_n^2(t)\,w(t)\,dt$$

and

$$\theta^2(2) = D^4 \int_a^b w(t)\, dt \int_{R^3} x^2 y^2 (\psi(\alpha(u); x, y) - \varphi(x)\, \varphi(y))\, du\, dx\, dy$$

$$+ 4D^2 \int_a^b r_n^2(t)\, w(t)\, dt \int_{R^3} xy(\psi(\alpha(u); x, y) - \varphi(x)\, \varphi(y))\, du\, dx\, dy$$

$$+ 2D^3 \int_a^b r_n(t)\, w(t)\, dt \int_{R^3} xy(x+y)(\psi(\alpha(u); x, y)$$

$$- \varphi(x)\, \varphi(y)\, du\, dx\, dy.$$

Wahba [19], Rice and Rosenblatt [15], and Ragozin [14] studied the behavior of $\int_a^b r_n^2(t)\, dt$. They showed that the rate at which $\int_a^b (g_{(n)}(t) - g(t))^2\, dt$ tends to zero is determined by the rate at which the Fourier coefficient of g decays, which depends on the smoothness of g periodically extended. Assuming that g is smooth they got that

$$\int_a^b (g_{(n)}(t) - g(t))^2\, dt = O(\lambda^2(n)). \tag{1.6}$$

Let $u_n(t) = g_{(n)}(t) - g(t)$ and $w(t) = 1$. If g is smooth (i.e., (1.6) holds) and $n\lambda^{9/4}(n) \to 0$, then by Theorem 2 we have

$$\{n(b-a)^{1/4}\, \lambda^{1/4}(n)\, I_n^*(p) - (D^2(b-a) + \int_a^b n\lambda^{1/4}(n)(g_{(n)}(t) - g(t))^2\, dt)\}/(\hat{\theta}^2(b-a)^{1/4}\, \lambda^{1/4}(n))^{1/2} \xrightarrow{\mathscr{D}} N(0, 1), \tag{1.7}$$

where

$$\hat{\theta}^2 = D^4(b-a) \int_{R^3} x^2 y^2 (\psi(\alpha(u); x, y) - \varphi(x)\, \varphi(y))\, du\, dx\, dy.$$

If we assume that $n\lambda^{17/8}(n) \to 0$, then (1.7) can be replaced by

$$\{n(b-a)^{1/4}\, \lambda^{1/4}(n)\, I_n^*(p) - D^2(b-a)\}/(\hat{\theta}^2(b-a)^{1/4}\, \lambda^{1/4}(n))^{1/2} \xrightarrow{\mathscr{D}} N(0, 1). \tag{1.8}$$

Similar arguments can be used to get the limit distribution of $\int_a^b |g_n(t) - g(t)|^p\, w(t)\, dt$, if $p \geqslant 2$ and $\lambda(n)$ is small enough.

Wahba [19] and Silverman [17] discussed the choice of the smoothing parameter λ. They suggested that the optimal λ should minimize $EI_n(2)$, the integrated mean squared error. Wahba [19] and Rice and Rosenblatt [15] showed that the optimal smoothing parameter λ must be proportional to $n^{-4/9}$. In this case we can apply Theorem 1 to get the limit distribution of the stochastic part of the smoothing spline. Combining (1.6) and Theorem 2 we can get a similar result for $I_n^*(p)$, $2 \leqslant p < \infty$, assuming that g is smooth. Cox [2] contains some asymptotic results on the bias of splines, which might be useful to get the limit distribution of $I_n^*(p)$, $1 \leqslant p < \infty$.

2. Proofs

First we need a few notations. Let

$$K^*(u) = K^*_{t,n}(u) = -(b-a)\, 2^{-3/2} r_{t,n} \exp(-2^{-1/2}|u|) \sin(2^{-1/2}|u| - \alpha_{t,n}),$$

where $\delta(t) = \min(t-a, b-t)$,

$$r_{t,n} \cos \alpha_{t,n} = 1 - 2 \sin(2^{1/2}\delta(t)(b-a)^{-1/4}(t)\, \lambda^{-1/4})$$

and

$$r_{t,n} \sin \alpha_{t,n} = 1 + 2 \cos(2^{1/2}\delta(t)(b-a)^{1/4}\, \lambda^{-1/4}).$$

Thus we have

$$6 - \sqrt{2} \leqslant r^2_{t,n} \leqslant 6 + 4\sqrt{2}. \tag{2.1}$$

We also define

$$x^* = \begin{cases} 2a - x, & \text{if } \ a \leqslant x < \dfrac{a+b}{2} \\[2mm] 2b - x, & \text{if } \ \dfrac{a+b}{2} < x \leqslant b. \end{cases} \tag{2.2}$$

Throughout this paper C_i stands for an absolute constant.

The following lemma is a special case of Theorem B of Silverman [17]. Silverman [17] did not assume that the design points are evenly spaced and so got a somewhat weaker result than what we need now. However, the proof of the following lemma follows immediately from the proof of Theorem B of Silverman [17].

Lemma 1. *We assume that* $\lambda(n) \to 0$ $(n \to \infty)$. *There are constants* C_1 *and* C_2 *such that*

$$\sup_{a \leqslant t, x \leqslant b} |G_n(t, x) - K_1(t, x)| \leqslant C_1 \left(\frac{1}{n\lambda^{1/2}} + \exp(-C_2(b-a)/\lambda^{1/4}) \right),$$

where $h = \lambda^{1/4}$ *and*

$$K_1(t, x) = \frac{1}{h}\{K((t-x)/h) + K^*((t-x^*)/h)\}.$$

Proof. Following the proof of Theorem B of Silverman [17], we get the exponential term $\exp(-C_2(b-a)/\lambda^{1/4})$ in the modified corollary of Silverman [17]. We assume the design points are evenly spaced and therefore the second integral in the definition of $G_{n,\lambda(t)}$ in (4.11) of Silverman [17] is zero for each $t > 0$. Thus we get that in this case $\sup_{(\lambda)} |G_{n,\lambda}| \leqslant C_3/(n\lambda^{1/4})$, which completes the sketch of the proof of Lemma 1.

Let

$$g_{n,1}(t) = \frac{1}{n}\sum_{i=1}^{n} K_1(t, x_i)\,\xi_i.$$

Using Lemma 1 we can estimate the difference between L_p-norms of $\hat{g}_n$ and $g_{n,1}$. The estimator $g_{n,1}$ is a kernel regression estimator, but the kernel depends on n as well as t.

LEMMA 2. *We assume that* $nh^2(n) \to \infty$, $h(n) \to 0$, $(n \to \infty)$, *and* C2, C5 *hold. Then, as* $n \to \infty$, *we have*

$$\int_a^b |(nh)^{1/2}\,\hat{g}_n(t) + r_n(t)|^p\, w(t)\,dt - \int_a^b |(nh)^{1/2}\, g_{n,1}(t) + r_n(t)|^p\, w(t)\,dt = o_P(h^{1/2}).$$

Proof. First we note

$$\gamma_n^2(t) = \operatorname{var}((nh)^{1/2}\,\hat{g}_n(t)) = \sigma^2\frac{h}{n}\sum_{i=1}^{n} G_n^2(t, x_i), \tag{2.3}$$

$$\gamma_{n,1}^2(t) = \operatorname{var}((nh)^{1/2}\, g_{n,1}(t)) = \sigma^2\frac{h}{n}\sum_{i=1}^{n} K_1^2(t, x_i) \tag{2.4}$$

and

$$\delta_{n,t}(t) = \operatorname{cov}((nh)^{1/2}\,\hat{g}_n(t), (nh)^{1/2}\, g_{n,1}(t)) = \sigma^2\frac{h}{n}\sum_{i=1}^{n} G_n(t, x_i)\, K_1(t, x_i). \tag{2.5}$$

We get from Lemma 1 that

$$\sup_{a \leqslant t, x \leqslant b} |G_n(t, x) - K_1(t, x)| = o(1), \tag{2.6}$$

which implies immediately that

$$|\gamma_n^2(t) - \gamma_{n,1}^2(t)| = o(1) \left\{ h + \frac{h}{n} \sum_{i=1}^{n} |K_1(t, x_i)| \right\} \tag{2.7}$$

and

$$|\delta_{n,1}(t) - \gamma_{n,1}^2(t)| = o(1) \frac{h}{n} \sum_{i=1}^{n} |K_1(t, x_i)|. \tag{2.8}$$

It follows from the definition of the design points that

$$\sup_{a \leqslant t \leqslant b} \left| \frac{h}{n} \sum_{i=1}^{n} K_1^2(t, x_i) - h \int_a^b K_1^2(t, x)\, dx \right| = O\left(\frac{1}{nh^2}\right). \tag{2.9}$$

Also,

$$\begin{aligned} h \int_a^b K_1^2(t, x)\, dx &= \frac{1}{h} \int_a^b K^2((t-x)/h)\, dx \\ &\quad + \frac{2}{h} \int_a^b K((t-x)/h)\, K^*((t-x^*)/h)\, dx \\ &\quad + \frac{1}{h} \int_a^b (K^*((t-x^*)/h))^2\, dx \\ &= R_n^{(1)}(t) + R_n^{(2)}(t) + R_n^{(3)}(t). \end{aligned} \tag{2.10}$$

It is easy to see that

$$R_n^{(1)}(t) = \int_{-(b-t)/h}^{(t-a)/h} K^2(u)\, du, \tag{2.11}$$

and therefore

$$\begin{aligned} &\left| R_n^{(1)}(t) - \int_{-\infty}^{\infty} K^2(u)\, du \right| \\ &\quad \leqslant C_4 \{\exp(-2^{1/2}(t-a)/h) + \exp(-2^{1/2}(b-t)/h\}. \end{aligned} \tag{2.12}$$

Next we note that

$$|R_n^{(2)}(t)| + |R_n^{(3)}(t)| \leqslant C_5 \frac{1}{h} \int_a^b \exp(-2^{-1/2}\, |t - x^*|/h)\, dx. \tag{2.13}$$

Using the definition of x^* we get

$$\frac{1}{h}\int_a^{(a+b)/2} \exp(-2^{-1/2}|t-x^*|/h)\,dx$$
$$=\frac{1}{h}\int_a^{(a+b)/2} \exp(-2^{-1/2}(t+x-2a)/h)\,dx$$
$$\leqslant C_6\exp(-2^{-1/2}(t-a)/h) \tag{2.14}$$

and a similar argument gives

$$\frac{1}{h}\int_{(a+b)/2}^{b} \exp(-2^{-1/2}|t-x^*|/h)\,dx \leqslant C_7\exp(-2^{-1/2}(b-t)/h). \tag{2.15}$$

Putting together (2.4), (2.10), 2.12)–(2.15) we obtain that

$$\sup_{a\leqslant t\leqslant b} |\gamma_{n,1}^2(t)| \leqslant C_8. \tag{2.16}$$

Similar calculations give that

$$\sup_{a\leqslant t\leqslant b} \frac{1}{n}\left|\sum_{i=1}^{n} K_1(t,x_i)\right| \leqslant C_9, \tag{2.17}$$

and therefore (2.7) and (2.8) imply

$$\sup_{a\leqslant t\leqslant b} |\gamma_n^2(t)-\gamma_{n,1}^2(t)| = o(h) \tag{2.18}$$

and

$$\sup_{a\leqslant t\leqslant b} |\delta_{n,1}(t)-\gamma_{n,1}^2(t)| = o(h). \tag{2.19}$$

Next we show that there is a positive constant C_{10} such that

$$\inf_{a\leqslant t\leqslant b} \gamma_{n,1}^2(t) \geqslant C_{10}. \tag{2.20}$$

By (2.10), (2.12)–(2.15) we can find two positive constants C_{11} and C_{12} such that

$$\inf_{a+C_{12}h\leqslant t\leqslant b-C_{12}h} \gamma_{n,1}^2(t) \geqslant C_{11}. \tag{2.21}$$

Applying (2.4) and (2.9) we get that

$$\lim_{n\to\infty} \inf_{a\leqslant t\leqslant a+C_{12}h} \gamma_{n,1}^2(t) = \lim_{n\to\infty} \inf_{0\leqslant t\leqslant C_{12}h} \sigma^2 h \int_a^b K_1^2(t,x)\,dx$$

$$\geqslant \lim_{n\to\infty} \inf_{a\leqslant t\leqslant a+C_{12}h} \frac{\sigma^2}{h} \int_a^{(a+b)/2} (K((t-x)/h) + K_{t,n}^*((t+x^*)/h))^2\,dx$$

$$= \sigma^2 \lim_{n\to\infty} \inf_{a\leqslant t\leqslant a+C_{12}h} \int_{-((a+b)/2-t)/h}^{(t-a)/h} (K(u) + K_{t,n}^*(-u+2(t-a)/h))^2\,du$$

$$= \sigma^2 \inf_{0\leqslant x\leqslant C_{12}} \int_{-\infty}^{x} (K(u) + K_x^*(-u+2x))^2\,du > 0, \tag{2.22}$$

where

$$K_x^*(u) = -2^{-3/2} r_x \exp(-2^{-1/2}|u|)\sin(2^{-1/2}|u| - \alpha_x),$$
$$r_x \cos\alpha_x = 1 - 2\sin(2^{1/2}x)$$
$$r_x \sin\alpha_x = 1 + 2\cos(2^{1/2}x).$$

A similar argument is working when $b - C_{12}h \leqslant t \leqslant b$, which also completes the proof of (2.20).

Let $\rho_{n,1}(t) = \operatorname{cor}((nh)^{1/2}\hat{g}_n(t), (nh)^{1/2} g_{n1}(t))$. It is immediate from (2.18), (2.19), and (2.20) that

$$\sup_{a\leqslant t\leqslant b} |1 - \rho_{n,1}(t)| = o(h). \tag{2.23}$$

We assume that $3 \leqslant r \leqslant v$. It is easy to check that

$$\sup_{a\leqslant t\leqslant b} \frac{1}{n} \sum_{i=1}^{n} |h^{1/2} K_1(t,x_i)|^r E|\xi_i|^r = O(h^{(2-r)/2}) \tag{2.24}$$

and by Lemma 1 we have

$$\sup_{a\leqslant t\leqslant b} \frac{1}{n} \sum_{i=1}^{n} |h^{1/2} G_n(t,x_i)|^r E|\xi_i|^r = O(h^{(2-r)/2}). \tag{2.25}$$

Using the assumption $nh^2 \to \infty$ $(n\to\infty)$ and (2.16), (2.18), (2.24), (2.25) we find two constants C_{13} and C_{14} such that

$$\sup_{a\leqslant t\leqslant b} \frac{1}{n} \sum_{i=1}^{n} \left|\frac{h^{1/2}}{\gamma_{n,1}(t)} K_1(t,x_i)\right|^r E|\xi_i|^r \leqslant C_{13} h^{(2-r)/2} \tag{2.26}$$

and

$$\sup_{a \leqslant t \leqslant b} \frac{1}{n} \sum_{i=1}^{n} \left| \frac{h^{1/2}}{\gamma_n(t)} G_n(t, x_i) \right|^r E\,|\xi_i|^r \leqslant C_{14} h^{(2-r)/2}. \tag{2.27}$$

Choosing $r > \max(3, 2p-2)$ in (2.26) and (2.27) we can apply Theorem 17.6 in Bhattacharya and Rao [1]. If N stands for a standard normal r.v. then we can write

$$\sup_{a \leqslant t \leqslant b} |E\,|(nh)^{1/2}(g_{n,1}(t))|^{2p-2} - (\gamma_{n,1}(t))^{2p-2}\, E\,|N|^{2p-2}| = O((nh)^{-1/2}), \tag{2.28}$$

which implies

$$\sup_{a \leqslant t \leqslant b} E\,|(nh)^{1/2}\, \hat{g}_n(t)|^{2p-2} \leqslant C_{15}. \tag{2.29}$$

Similar arguments give

$$\sup_{a \leqslant t \leqslant b} E\,|(nh)^{1/2}\, \hat{g}_n(t)|^{2p-2} \leqslant C_{16}. \tag{2.30}$$

The Cauchy inequality yields

$$\begin{aligned} &E\,|\,|(nh)^{1/2}\, \hat{g}_n(t) + r_n(t)|^p - |(nh)^{1/2}\, g_{n,1}(t) + r_n(t)|^p| \\ &\quad\leqslant 2^{p-1}(E\{|(nh)^{1/2}\, \hat{g}_n(t) + r_n(t)|^{2p-2} + |(nh)^{1/2}\, g_{n,1}(t) + r_n(t)|^{2p-2}\}) \\ &\quad\quad \times (E((nh)^{1/2}\, (\hat{g}_n(t) - g_{n,1}(t)))^2)^{1/2}. \end{aligned} \tag{2.31}$$

We get immediately from (2.23) that

$$\sup_{a \leqslant t \leqslant b} E((nh)^{1/2}\, (\hat{g}_n(t) - g_{n,1}(t)))^2 = o(h). \tag{2.32}$$

Thus we proved

$$\begin{aligned} \int_a^b E\,|\,|(nh)^{1/2}\, \hat{g}_n(t) + r_n(t)|^p \\ - |(nh)^{1/2}\, g_{n,1}(t) + r_n(t)|^p|\, w(t)\, dt = o(h^{1/2}), \end{aligned} \tag{2.33}$$

which implies Lemma 2.

According to the following lemma we can assume without loss of generality that the original observations are normal r.v.'s.

LEMMA 3. *We assume C1, C2, and C5 hold. Then we can define a sequence of Wiener processes* $\{W_n(u), -\infty < u < \infty\}$ *such that*

$$\int_a^b |(nh)^{1/2} g_{n,1}(t) + r_n(t)|^p w(t)\, dt - \int_a^b |\Gamma_{n,1}(t) + r_n(t)|^p\, dt = o_P(h^{1/2}),$$

where

$$\Gamma_{n,1}(t) = \left(\frac{h\sigma^2}{b-a}\right)^{1/2} \int_a^b K_1(t, x)\, dW_n(x).$$

Proof. Introducing

$$S(x) = \sum_{1 \leqslant i \leqslant x} \xi_i, \qquad S(x) = 0,\, 0 \leqslant x \leqslant 1,$$

one can write

$$(nh)^{1/2} g_{n,1}(t) = \left(\frac{h}{n}\right)^{1/2} \int_{(0,n]} K_1\left(t, a + \frac{x}{n}(b-a)\right) dS(x). \tag{2.34}$$

Komlós *et al.* [10, 11] and Major [13] constructed a Wiener process $\{W(x), x \geqslant 0\}$ such that

$$\sup_{0 \leqslant x \leqslant n} |S(x) - \sigma W(x)| = o(n^{1/v}) \qquad \text{a.s.} \tag{2.35}$$

Integration by parts and (2.35) yield

$$\begin{aligned}
&\sup_{a \leqslant t \leqslant b} \left| (nh)^{1/2} g_{n,1}(t) - \left(\frac{h}{n}\right)^{1/2} \sigma \int_0^n K_1\left(t, a + \frac{x}{n}(b-a)\right) dW(x) \right| \\
&\quad = o_P(n^{1/v}/(nh)^{1/2}) \\
&\qquad + o_P\left(\left(\frac{h}{n}\right)^{1/2} \int_0^n \left| \frac{\partial}{\partial x} K_1\left(t, a + \frac{x}{n}(b-a)\right) \right| dx \right) \\
&\quad = o_P(n^{1/v}/(nh)^{1/2}).
\end{aligned} \tag{2.36}$$

The scale and shift transformation of the Wiener process implies

$$\begin{aligned}
&\left\{ \left(\frac{h}{n}\right)^{1/2} \sigma \int_0^n K_1\left(t, a + \frac{x}{n}(b-a)\right) dW(x),\, a \leqslant t \leqslant b \right\} \\
&\quad \stackrel{\mathscr{D}}{=} \left\{ \left(\frac{h}{n}\right)^{1/2} \sigma \int_a^b K_1(t, u)\, dW\left(n \frac{u-a}{b-a}\right),\, a \leqslant t \leqslant b \right\} \\
&\quad \stackrel{\mathscr{D}}{=} \left\{ \left(\frac{h}{b-a}\right)^{1/2} \sigma \int_a^b K_1(t, u)\, dW(u),\, a \leqslant t \leqslant b \right\},
\end{aligned} \tag{2.37}$$

which also completes the proof of Lemma 3.

The distribution of the Wiener processes W_n does not depend on n, therefore it is enough to study the L_p-norm of

$$\Gamma_1(t)=\left(\frac{h}{b-a}\right)^{1/2}\sigma\int_a^b K_1(t,u)\,dW(u).$$

We define

$$\Gamma_2(t)=\left(\frac{\sigma^2}{h(b-a)}\right)^{1/2}\int_a^b K((t-u)/h)\,dW(u),$$

and show that the difference between Γ_1 and Γ_2 is small.

LEMMA 4. *If* C1 *and* C2 *hold, then we have*

$$\int_a^b |\Gamma_1(t)+r_n(t)|^p\,w(t)\,dt$$
$$-\int_a^b |\Gamma_2(t)+r_n(t)|^p\,w(t)\,dt=o_P(h^{1/2}).$$

Proof. Let

$$\gamma_{n,3}^2(t)=\operatorname{var}\Gamma_1(t)=\frac{\sigma^2 h}{b-a}\int_a^b K_1^2(t,u)\,du,$$
$$\gamma_{n,4}^2(t)=\operatorname{var}\Gamma_2(t)=\frac{\sigma^2}{h(b-a)}\int_a^b K^2((t-u)/h)\,du$$

and

$$\delta_{n,2}(t)=\operatorname{cov}(\Gamma_1(t),\Gamma_2(t))=\frac{\sigma^2}{b-a}\int_a^b K((t-u)/h)\,K_1(t,u)\,du.$$

By definition of K_1 we have

$$\gamma_{n,3}^2(t)=\gamma_{n,4}^2(t)+\frac{2\sigma^2}{h(b-a)}\int_a^b K((t-u)/h)\,K^*((t-u^*)/h)\,du$$
$$+\frac{\sigma^2}{h(b-a)}\int_a^b (K^*((t-u^*)/h))^2\,du. \tag{2.38}$$

Similarly to (2.14) we obtain that

$$|\gamma_{n,3}^2(t)-\gamma_{n,4}^2(t)|\leqslant C_{17}\{\exp(-2^{-1/2}(t-a)/h)$$
$$+\exp(-2^{-1/2}(b-t)/h)\} \tag{2.39}$$

and

$$|\delta_{n,2}(t)-\gamma_{n,4}^2(t)| \leqslant C_{18}\{\exp(-2^{-1/2}(t-a)/h) + \exp(-2^{-1/2}(b-t)/h\}. \tag{2.40}$$

The processes Γ_1 and Γ_2 are Gaussian and therefore

$$E\,|\Gamma_1(t)|^r = \gamma_{n,3}^{r/2} E\,|N|^r, \tag{2.41}$$

$$E\,|\Gamma_2(t)|^r = \gamma_{n,4}^{r/2} E\,|N|^r \tag{2.42}$$

for all $r>0$, where N is a standard normal r.v. Using (2.39) we can find two constants C_{19} and C_{20} such that

$$\sup_{a\leqslant t\leqslant b} \gamma_{n,3}^2(t) \leqslant C_{19} \tag{2.43}$$

and

$$\sup_{a\leqslant t\leqslant b} \gamma_{n,4}^2(t) \leqslant C_{20}. \tag{2.44}$$

Thus we get from (2.41)–(2.44) that

$$\int_a^{a+h^{3/4}} E\,|\,|\Gamma_1(t)+r_n(t)|^p - |\Gamma_2(t)+r_n(t)|^p|\, w(t)\,dt = o(h^{1/2}) \tag{2.45}$$

and

$$\int_{b-h^{3/4}}^{b} E\,|\,|\Gamma_1(t)+r_n(t)|^p - |\Gamma_2(t)+r_n(t)|^p|\, w(t)\,dt = o(h^{1/2}). \tag{2.46}$$

It follows from (2.39) and (2.4) that

$$\inf_{a+h^{3/4}\leqslant t\leqslant b-h^{3/4}} \gamma_{n,3}^2(t) > C_{21}, \tag{2.47}$$

$$\inf_{a+h^{3/4}\leqslant t\leqslant b-h^{3/4}} \gamma_{n,4}^2(t) > C_{22} \tag{2.48}$$

and

$$\sup_{a+h^{3/4}\leqslant t\leqslant b-h^{3/4}} |\delta_{n,2}(t)-\gamma_{n,4}^2(t)| = o(h). \tag{2.49}$$

Introducing $\rho_{n,2}(t)=\mathrm{cor}(\Gamma_1(t), \Gamma_2(t))$ we have by (2.47)–(2.49) that

$$\sup_{a+h^{3/4}\leqslant t\leqslant b-h^{3/4}} |1-\rho_{n,2}(t)| = o(h). \tag{2.50}$$

Let $a+h^{3/4}\leqslant t\leqslant b-h^{3/4}$. Applying the Cauchy inequality and (2.41), (2.42), (2.49) we obtain that

$$\begin{aligned} E\,|\,|\Gamma_1(t)+r_n(t)|^p-|\Gamma_2(t)+r_n(t)|^p| \\ \leqslant 2^{p-1}(E\{|\Gamma_1(t)+r_n(t)|^{2p-2}+|\Gamma_2(t)+r_n(t)|^{2p-2}\})^{1/2} \\ \times(E(\Gamma_1(t)-\Gamma_2(t))^2)^{1/2}=o(h^{1/2}) \end{aligned} \tag{2.51}$$

uniformly on $[a+h^{3/4}, b-h^{3/4}]$. Now Lemma 4 follows from (2.45), (2.46), and (2.51).

Let

$$\Gamma_3(t)=\frac{\sigma}{(h(b-a))^{1/2}}\int_{-\infty}^{\infty}K((t-u)/h)\,dW(u).$$

Next we estimate the difference between Γ_2 and Γ_3.

LEMMA 5. *If* C1 *and* C2 *hold, then as* $n\to\infty$, *we have*

$$\int_a^b|\Gamma_2(t)+r_n(t)|^p\,w(t)\,dt-\int_a^b|\Gamma_3(t)+r_n(t)|^p\,w(t)\,dt=o_P(h^{1/2}).$$

Proof. We follow the proof of Lemma 4. Let

$$\gamma_{n,5}^2(t)=\operatorname{var}\Gamma_3(t)=\frac{\sigma^2}{h(b-a)}\int_{-\infty}^{\infty}K^2((t-u)/h)\,du=\frac{\sigma^2}{b-a}\int_{-\infty}^{\infty}K^2(u)\,du$$

and

$$\delta_{n,3}(t)=\operatorname{cov}(\Gamma_2(t),\Gamma_3(t))=\frac{\sigma^2}{h(b-a)}\int_a^b K^2((t-u)/h)\,du.$$

An elementary calculation shows that

$$\begin{aligned} |\gamma_{n,4}^2(t)-\gamma_{n,5}^2(t)|\leqslant C_{23}\{\exp(-2^{1/2}(t-a)/h) \\ +\exp(-2^{1/2}(b-t)/h)\} \end{aligned} \tag{2.52}$$

and

$$\begin{aligned} |\delta_{n,3}(t)-\gamma_{n,5}^2(t)|\leqslant C_{24}\{\exp(-2^{1/2}(t-a)/h) \\ +\exp(-2^{1/2}(b-t)/h)\}. \end{aligned} \tag{2.53}$$

The bounds in (2.52) and (2.53) appeared earlier in (2.39) and (2.40). Thus continuing along the lines of the proof of Lemma 4 we get

$$\int_a^b E\,|\,|\Gamma_2(t)+r_n(t)|^p-|\Gamma_3(t)+r_n(t)|^p|\,w(t)\,dt=o(h^{1/2}),$$

which also completes the proof of Lemma 5.

Central limit theorems for L_p-norms of a kernel-transformed Wiener process were obtained by Csörgő and Horváth [4] and Horváth [9]. They assumed that the kernel has a finite support and therefore their result cannot be used immediately in our case. First we use the truncated kernel

$$K_A(u)=\begin{cases}K(u), & \text{if } |u|\leqslant A\\ 0, & \text{if } |u|>A,\end{cases}$$

and then we show that the central limit remains true when $A\to\infty$. Let

$$\Gamma_A(t)=\frac{\sigma}{(h(b-a))^{1/2}}\int_{-\infty}^{\infty}K_A((t-u)/h)\,dW(u)$$

and define

$$D_A^2=\frac{\sigma^2}{b-a}\int_{-\infty}^{\infty}K_A^2(u)\,du,$$

$$\alpha_A(s)=\int_{-\infty}^{\infty}K_A(t)\,K_A(t+s)\,dt\bigg/\int_{-\infty}^{\infty}K_A^2(t)\,dt,$$

$$m_A=E\int_a^b|\Gamma_A(t)+r_n(t)|^p\,w(t)\,dt$$

$$=\int_{-\infty}^{\infty}\int_a^b|D_Ax+r_n(t)|^p\,w(t)\,\varphi(x)\,dt\,dx,$$

$$\theta^2(\beta;U,V)=\int_{R^3}\int_a^b|UVxy+r_n(t)(Ux+Vy)+r_n^2(t)|^p\,w^2(t)(\psi(\beta(u);x,y)$$
$$-\varphi(x)\,\varphi(y))\,dt\,dx\,dy\,du,$$

and

$$\theta_A^2=\theta^2(\alpha_A;D_A,D_A).$$

Now we can state the result of Csörgő and Horváth [4].

LEMMA 6. *We assume that* C1, C2, and C3 *hold*, $A > 0$. *Then as* $n \to \infty$, *we have*

$$\left\{\int_a^b |\Gamma_A(t) + r_n(t)|^p w(t)\,dt - m_A\right\}\Big/(\theta_A h^{1/2}) \xrightarrow{\mathscr{D}} N(0, 1),$$

where $N(0, 1)$ *stands for a standard normal r.v.*

The following result shows that lemma 6 remains true when $A = \infty$.

LEMMA 7. *We assume that* C1, C2, *and* C3 *hold. Then, as* $n \to \infty$, *we have*

$$\left\{\int_a^b |\Gamma_3(t) + r_n(t)|^p w(t)\,dt - m\right\}\Big/(\theta h^{1/2}) \xrightarrow{\mathscr{D}} N(0, 1), \qquad (2.54)$$

where $N(0, 1)$ *stands for a standard normal r.v.*

Proof. Let

$$\eta_A = \int_a^b (|\Gamma_3(t) + r_n(t)|^p - |\Gamma_A(t) + r_n(t)|^p)\,w(t)\,dt - (m - m_A),$$

and

$$\alpha_{1,A}(s) = \int_{-\infty}^{\infty} K_A(u)\,K(u+s)\,du\left(\int_{-\infty}^{\infty} K_A^2(u)\,du \int_{-\infty}^{\infty} K_1^2(u)\,du\right)^{-1/2}.$$

Lengthy but elementary calculations give that

$$|E\eta_A^2/h - (\theta^2 + \theta_A^2 - 2\theta^2(\alpha_{1,A}; D, D_A))| = o(1), \qquad (n \to \infty),$$

$$\lim_{A \to \infty} (\theta^2 + \theta_A^2 - 2\theta^2(\alpha_{1,A}; D, A_A)) = 0,$$

and

$$\lim_{A \to \infty} \theta^2/\theta_A^2 = 1.$$

Now Lemma 6 implies (2.54).

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